$= (F_1, F_2, F_3)(x,y,z)$ $= (G_1, G_2, G_3)(x,y,z)$

Έστω διανυσματικά πεδία $\bar{F}(x,y,z)$, $\bar{G}(x,y,z)$ και βαθμωτά πεδία $f(x,y,z)$, $g(x,y,z)$, παραγωγίσιμα.

Να δειχθούν τα ακόλουθα:

(i) $\nabla(f+g) = \nabla f + \nabla g$, $\nabla(fg) = f\nabla g + g\nabla f$

Είναι: $\nabla(f+g) = \left(\frac{\partial}{\partial x}(f+g), \frac{\partial}{\partial y}(f+g), \frac{\partial}{\partial z}(f+g)\right)$

$= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$+ \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = \nabla f + \nabla g.$

$\nabla(fg) = \left(\frac{\partial}{\partial x}(fg), \frac{\partial}{\partial y}(fg), \frac{\partial}{\partial z}(fg)\right)$

$= \left(g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}, g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}, g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z}\right)$

$= g\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) + f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$

$= g\nabla f + f\nabla g.$

ii) $\nabla \cdot (\bar{F} + \bar{G}) = \nabla \cdot \bar{F} + \nabla \cdot \bar{G}$

Είναι, $\nabla \cdot (\bar{F} + \bar{G}) = \frac{\partial}{\partial x}(\bar{F}+\bar{G})_1 + \frac{\partial}{\partial y}(\bar{F}+\bar{G})_2$

$+ \frac{\partial}{\partial z}(\bar{F}+\bar{G})_3 = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial x}G_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial y}G_2$

$+ \frac{\partial}{\partial z}F_3 + \frac{\partial}{\partial z}G_3 = \nabla \cdot \bar{F} + \nabla \cdot \bar{G}$

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iii) $\nabla \times (\bar{F} + \bar{G}) = \nabla \times \bar{F} + \nabla \times \bar{G}$

Είναι, $\nabla \times (\bar{F} + \bar{G}) = \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (\bar{F}+\bar{G})_1 & (\bar{F}+\bar{G})_2 & (\bar{F}+\bar{G})_3 \end{pmatrix}$

$= \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix} + \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ G_1 & G_2 & G_3 \end{pmatrix}$

$= \nabla \times \bar{F} + \nabla \times \bar{G}$, με $\{e_1, e_2, e_3\}$ συνήθης βάση του $\mathbb{R}^3$.

iv) $\nabla \cdot (f\bar{F}) = (\nabla f) \cdot \bar{F} + f(\nabla \cdot \bar{F})$

Είναι, $\nabla \cdot (f\bar{F}) = \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial z}(fF_3)$

$= \frac{\partial f}{\partial x}F_1 + \frac{\partial F_1}{\partial x}f + \frac{\partial f}{\partial y}F_2 + \frac{\partial F_2}{\partial y}f + \frac{\partial f}{\partial z}F_3 + \frac{\partial F_3}{\partial z}f$

$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (F_1, F_2, F_3) + f\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$

$= (\nabla f) \cdot \bar{F} + f(\nabla \cdot \bar{F})$.

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v) $\nabla \times (f\bar{F}) = (\nabla f) \times \bar{F} + f(\nabla \times \bar{F})$

Είναι, $\nabla \times (f\bar{F}) = \det\begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fF_1 & fF_2 & fF_3 \end{pmatrix}$

$= \left(\frac{\partial}{\partial y}(fF_3) - \frac{\partial}{\partial z}(fF_2), \frac{\partial}{\partial z}(fF_1) - \frac{\partial}{\partial x}(fF_3), \frac{\partial}{\partial x}(fF_2) - \frac{\partial}{\partial y}(fF_1)\right) = \left(\frac{\partial f}{\partial y}F_3 + \frac{\partial F_3}{\partial y}f - \frac{\partial f}{\partial z}F_2 - \frac{\partial F_2}{\partial z}f,\right.$

$\frac{\partial f}{\partial z}F_1 + \frac{\partial F_1}{\partial z}f - \frac{\partial f}{\partial x}F_3 - \frac{\partial F_3}{\partial x}f, \frac{\partial f}{\partial x}F_2 + \frac{\partial F_2}{\partial x}f - \frac{\partial f}{\partial y}F_1$

$\left. - \frac{\partial F_1}{\partial y}f\right) = \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2, \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3, \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right)$

$$+ f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} + f \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix}$$

$= (\nabla f) \times \bar{F} + f(\nabla \times \bar{F})$, όπου $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ η συνήθης βάση του $\mathbb{R}^3$

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vi) $\nabla \cdot (\bar{F} \times \bar{G}) = \bar{G} \cdot (\nabla \times \bar{F}) - \bar{F} \cdot (\nabla \times \bar{G})$

Είναι $\nabla \cdot (\bar{F} \times \bar{G}) = \det \begin{pmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{pmatrix}$

$$= \frac{\partial}{\partial x}(F_2 G_3) - \frac{\partial}{\partial x}(F_3 G_2) + \frac{\partial}{\partial y}(F_3 G_1) - \frac{\partial}{\partial y}(F_1 G_3) + \frac{\partial}{\partial z}(F_1 G_2)$$

$$- \frac{\partial}{\partial z}(F_2 G_1) = \frac{\partial F_2}{\partial x} G_3 + \frac{\partial G_3}{\partial x} F_2 - \frac{\partial F_3}{\partial x} G_2 - \frac{\partial G_2}{\partial x} F_3$$

$$+ \frac{\partial F_3}{\partial y} G_1 + \frac{\partial G_1}{\partial y} F_3 - \frac{\partial F_1}{\partial y} G_3 - \frac{\partial G_3}{\partial y} F_1 + \frac{\partial F_1}{\partial z} G_2$$

$$+ \frac{\partial G_2}{\partial z} F_1 - \frac{\partial F_2}{\partial z} G_1 - \frac{\partial G_1}{\partial z} F_2 = \Big[(G_1, G_2, G_3) \cdot$$

$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \Big] - \Big[(F_1, F_2, F_3) \cdot$$

$$\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}, \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}, \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right)$$

$$= \bar{G} \cdot \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix} - \bar{F} \cdot \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ G_1 & G_2 & G_3 \end{pmatrix}$$

$= \bar{G} \cdot (\nabla \times \bar{F}) - \bar{F} \cdot (\nabla \times \bar{G})$, όπου $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ η συνήθης βάση του $\mathbb{R}^3$

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vii) $\nabla \times (\bar{F} \times \bar{G}) = (\bar{G} \cdot \nabla)\bar{F} - \bar{G}(\nabla \cdot \bar{F}) - (\bar{F} \cdot \nabla)\bar{G} + \bar{F}(\nabla \cdot \bar{G})$

<u>Είναι</u>, $\bar{F} \times \bar{G} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = (F_2 G_3 - F_3 G_2, F_3 G_1 - F_1 G_3, F_1 G_2 - F_2 G_1)$

, οπότε $\nabla \times (\bar{F} \times \bar{G}) = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 G_3 - F_3 G_2 & F_3 G_1 - F_1 G_3 & F_1 G_2 - F_2 G_1 \end{vmatrix}$

$$= \Big(\frac{\partial F_1}{\partial y} G_2 + \frac{\partial G_2}{\partial y} F_1 - \frac{\partial F_2}{\partial y} G_1 - \frac{\partial G_1}{\partial y} F_2 - \frac{\partial F_3}{\partial z} G_1 - \frac{\partial G_1}{\partial z} F_3$$
$$+ \frac{\partial F_1}{\partial z} G_3 + \frac{\partial G_3}{\partial z} F_1, \frac{\partial F_2}{\partial z} G_3 + \frac{\partial G_3}{\partial z} F_2 - \frac{\partial F_3}{\partial z} G_2 - \frac{\partial G_2}{\partial z} F_3$$
$$- \frac{\partial F_1}{\partial x} G_2 - \frac{\partial G_2}{\partial x} F_1 + \frac{\partial F_2}{\partial x} G_1 + \frac{\partial G_1}{\partial x} F_2, \frac{\partial F_3}{\partial x} G_1 + \frac{\partial G_1}{\partial x} F_3$$

$$-\frac{\partial F_1}{\partial x}G_3 - \frac{\partial G_3}{\partial x}F_1 - \frac{\partial F_2}{\partial y}G_3 - \frac{\partial G_3}{\partial y}F_2 + \frac{\partial F_3}{\partial y}G_2$$

$$+\frac{\partial G_2}{\partial y}F_3\Big)$$ (1). Αναπτύσσω τώρα το 2ο μέλος της ζητούμενης σχέσης, κι έχω:

$$(\bar{G}\cdot\nabla)\bar{F} - \bar{G}(\nabla\cdot\bar{F}) - (\bar{F}\cdot\nabla)\bar{G} + \bar{F}(\nabla\cdot\bar{G}) =$$

$$=\Big[(G_1,G_2,G_3)\cdot\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\Big)\Big](F_1,F_2,F_3) - (G_1,G_2,G_3)\Big[\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\Big)\cdot(F_1,F_2,F_3)\Big] - \Big[(F_1,F_2,F_3)\cdot\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\Big)\Big](G_1,G_2,G_3)$$

$$+(F_1,F_2,F_3)\Big[\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\Big)\cdot(G_1,G_2,G_3)\Big] =$$

$$\Big(G_1\frac{\partial}{\partial x}+G_2\frac{\partial}{\partial y}+G_3\frac{\partial}{\partial z}\Big)(F_1,F_2,F_3) - (G_1,G_2,G_3)\Big(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\Big)$$

$$-\Big(F_1\frac{\partial}{\partial x}+F_2\frac{\partial}{\partial y}+F_3\frac{\partial}{\partial z}\Big)(G_1,G_2,G_3) + (F_1,F_2,F_3)\Big(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}+\frac{\partial G_3}{\partial z}\Big)$$

$$=\Big(\cancel{G_1\frac{\partial F_1}{\partial x}} + G_2\frac{\partial F_1}{\partial y} + G_3\frac{\partial F_1}{\partial z} - \cancel{G_1\frac{\partial F_1}{\partial x}} - G_1\frac{\partial F_2}{\partial y} - G_1\frac{\partial F_3}{\partial z}$$

$$-\cancel{F_1\frac{\partial G_1}{\partial x}} - F_2\frac{\partial G_1}{\partial y} - F_3\frac{\partial G_1}{\partial z} + \cancel{F_1\frac{\partial G_1}{\partial x}} + F_1\frac{\partial G_2}{\partial y} + F_1\frac{\partial G_3}{\partial z},$$

$$G_1\frac{\partial F_2}{\partial x} + \cancel{G_2\frac{\partial F_2}{\partial y}} + G_3\frac{\partial F_2}{\partial z} - G_2\frac{\partial F_1}{\partial x} - \cancel{G_2\frac{\partial F_2}{\partial y}} - G_2\frac{\partial F_3}{\partial z}$$

$$-F_1\frac{\partial G_2}{\partial x} - \cancel{F_2\frac{\partial G_2}{\partial y}} - F_3\frac{\partial G_2}{\partial z} + F_2\frac{\partial G_1}{\partial x} + \cancel{F_2\frac{\partial G_2}{\partial y}} + F_2\frac{\partial G_3}{\partial z},$$

$$G_1\frac{\partial F_3}{\partial x} + G_2\frac{\partial F_3}{\partial y} + G_3\frac{\partial F_3}{\partial z} - G_3\frac{\partial F_1}{\partial x} - G_3\frac{\partial F_2}{\partial y} - G_3\frac{\partial F_3}{\partial z}$$

$$- F_1 \frac{\partial G_3}{\partial x} - F_2 \frac{\partial G_3}{\partial y} - \cancel{F_3 \frac{\partial G_3}{\partial z}} + F_3 \frac{\partial G_1}{\partial x} + F_3 \frac{\partial G_2}{\partial y} + \cancel{F_3 \frac{\partial G_3}{\partial z}} \Bigg) \quad (2)$$

Από (1), (2) τα δυο μέλη είναι ίσα. □

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viii) $\nabla(\bar{F}\cdot\bar{G}) = (\bar{G}\cdot\nabla)\bar{F} + (\bar{F}\cdot\nabla)\bar{G} + \bar{G}\times(\bar{\nabla}\times\bar{F})$
$+ \bar{F}\times(\nabla\times\bar{G})$

Είναι $\nabla(\bar{F}\cdot\bar{G}) = \nabla\big((F_1,F_2,F_3)\cdot(G_1,G_2,G_3)\big)$

$$= \nabla(F_1G_1 + F_2G_2 + F_3G_3) = \Big(\frac{\partial}{\partial x}(F_1G_1 + F_2G_2 + F_3G_3),$$
$$\frac{\partial}{\partial y}(F_1G_1 + F_2G_2 + F_3G_3), \frac{\partial}{\partial z}(F_1G_1 + F_2G_2 + F_3G_3)\Big)$$
$$= \Big(\frac{\partial F_1}{\partial x}G_1 + \frac{\partial G_1}{\partial x}F_1 + \frac{\partial F_2}{\partial x}G_2 + \frac{\partial G_2}{\partial x}F_2 + \frac{\partial F_3}{\partial x}G_3 + \frac{\partial G_3}{\partial x}F_3,$$
$$\frac{\partial F_1}{\partial y}G_1 + \frac{\partial G_1}{\partial y}F_1 + \frac{\partial F_2}{\partial y}G_2 + \frac{\partial G_2}{\partial y}F_2 + \frac{\partial F_3}{\partial y}G_3 + \frac{\partial G_3}{\partial y}F_3,$$
$$\frac{\partial F_1}{\partial z}G_1 + \frac{\partial G_1}{\partial z}F_1 + \frac{\partial F_2}{\partial z}G_2 + \frac{\partial G_2}{\partial z}F_2 + \frac{\partial F_3}{\partial z}G_3 + \frac{\partial G_3}{\partial z}F_3\Big) \quad (1)$$

Αναπτύσσω τώρα το δεύτερο μέλος της ζητούμενης ισότητας, κι έχω:

$$(\bar{G}\cdot\nabla)\bar{F} + (\bar{F}\cdot\nabla)\bar{G} + \bar{G}\times(\nabla\times\bar{F}) + \bar{F}\times(\nabla\times\bar{G}) =$$

$$\left[(G_1,G_2,G_3)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\right](F_1,F_2,F_3) + \left[(F_1,F_2,F_3)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\right.$$

$$\Big](G_1,G_2,G_3) + (G_1,G_2,G_3)\times \overline{\det}\begin{pmatrix}\bar{e}_1 & \bar{e}_2 & \bar{e}_3\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_1 & F_2 & F_3\end{pmatrix}$$

$$+ (F_1,F_2,F_3)\times \overline{\det}\begin{pmatrix}\bar{e}_1 & \bar{e}_2 & \bar{e}_3\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ G_1 & G_2 & G_3\end{pmatrix} =$$

$$\left(G_1\frac{\partial}{\partial x} + G_2\frac{\partial}{\partial y} + G_3\frac{\partial}{\partial z}\right)(F_1,F_2,F_3) + \left(F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}\right)$$

$$(G_1,G_2,G_3) + (G_1,G_2,G_3)\times\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$+ (F_1,F_2,F_3)\times\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}, \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}, \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)$$

$$= \left(G_1\frac{\partial F_1}{\partial x} + G_2\frac{\partial F_1}{\partial y} + G_3\frac{\partial F_1}{\partial z}, G_1\frac{\partial F_2}{\partial x} + G_2\frac{\partial F_2}{\partial y} + G_3\frac{\partial F_2}{\partial z},\right.$$

$$\left. G_1\frac{\partial F_3}{\partial x} + G_2\frac{\partial F_3}{\partial y} + G_3\frac{\partial F_3}{\partial z}\right) + \left(F_1\frac{\partial G_1}{\partial x} + F_2\frac{\partial G_1}{\partial y} + F_3\frac{\partial G_1}{\partial z},\right.$$

$$\left. F_1\frac{\partial G_2}{\partial x} + F_2\frac{\partial G_2}{\partial y} + F_3\frac{\partial G_2}{\partial z}, F_1\frac{\partial G_3}{\partial x} + F_2\frac{\partial G_3}{\partial y} + F_3\frac{\partial G_3}{\partial z}\right)$$

$$+ \left(F_2\frac{\partial G_2}{\partial x} - F_2\frac{\partial G_1}{\partial y} - F_3\frac{\partial G_1}{\partial z} + F_3\frac{\partial G_3}{\partial x}, F_3\frac{\partial G_3}{\partial y} - F_3\frac{\partial G_2}{\partial z}\right.$$

$$\left. - F_1\frac{\partial G_2}{\partial x} + F_1\frac{\partial G_1}{\partial y}, F_1\frac{\partial G_1}{\partial z} - F_1\frac{\partial G_3}{\partial x} - F_2\frac{\partial G_3}{\partial y} + F_2\frac{\partial G_2}{\partial z}\right)$$

$$+ \left(G_2\frac{\partial F_2}{\partial x} - G_2\frac{\partial F_1}{\partial y} - G_3\frac{\partial F_1}{\partial z} + G_3\frac{\partial F_3}{\partial x}, G_3\frac{\partial F_3}{\partial y} - G_3\frac{\partial F_2}{\partial z}\right.$$

$$\left. - G_1\frac{\partial F_2}{\partial x} + G_1\frac{\partial F_1}{\partial y}, G_1\frac{\partial F_1}{\partial z} - G_1\frac{\partial F_3}{\partial x} - G_2\frac{\partial F_3}{\partial y} + G_2\frac{\partial F_2}{\partial z}\right)$$

$$= \Big(G_1 \frac{\partial F_1}{\partial x} + \cancel{G_2 \frac{\partial F_1}{\partial y}} + \cancel{G_3 \frac{\partial F_1}{\partial z}} + F_1 \frac{\partial G_1}{\partial x} + \cancel{F_2 \frac{\partial G_1}{\partial y}} + \cancel{F_3 \frac{\partial G_1}{\partial z}}$$

$$+ F_2 \frac{\partial G_2}{\partial x} - \cancel{F_2 \frac{\partial G_1}{\partial y}} - \cancel{F_3 \frac{\partial G_1}{\partial z}} + F_3 \frac{\partial G_3}{\partial x} + G_2 \frac{\partial F_2}{\partial x} - \cancel{G_2 \frac{\partial F_1}{\partial y}}$$

$$- \cancel{G_3 \frac{\partial F_1}{\partial z}} + G_3 \frac{\partial F_3}{\partial x} \,,\, \cancel{G_1 \frac{\partial F_2}{\partial x}} + G_2 \frac{\partial F_2}{\partial y} + \cancel{G_3 \frac{\partial F_2}{\partial z}} + \cancel{F_1 \frac{\partial G_2}{\partial x}}$$

$$+ F_2 \frac{\partial G_2}{\partial y} + \cancel{F_3 \frac{\partial G_2}{\partial z}} + F_3 \frac{\partial G_3}{\partial y} - \cancel{F_3 \frac{\partial G_2}{\partial z}} - \cancel{F_1 \frac{\partial G_2}{\partial x}} + F_1 \frac{\partial G_1}{\partial y}$$

$$+ G_3 \frac{\partial F_3}{\partial y} - \cancel{G_3 \frac{\partial F_2}{\partial z}} - \cancel{G_1 \frac{\partial F_2}{\partial x}} + G_1 \frac{\partial F_1}{\partial y} \,,\, \cancel{G_1 \frac{\partial F_3}{\partial x}} + \cancel{G_2 \frac{\partial F_3}{\partial y}}$$

$$+ G_3 \frac{\partial F_3}{\partial z} + \cancel{F_1 \frac{\partial G_3}{\partial x}} + \cancel{F_2 \frac{\partial G_3}{\partial y}} + F_3 \frac{\partial G_3}{\partial z} + F_1 \frac{\partial G_1}{\partial z} - \cancel{F_1 \frac{\partial G_3}{\partial x}}$$

$$- \cancel{F_2 \frac{\partial G_3}{\partial y}} + F_2 \frac{\partial G_2}{\partial z} + G_1 \frac{\partial F_1}{\partial z} - \cancel{G_1 \frac{\partial F_3}{\partial x}} - \cancel{G_2 \frac{\partial F_3}{\partial y}} + G_2 \frac{\partial F_2}{\partial z} \Big) \quad (2)$$

Από (1), (2) τα δυο μέλη είναι ίσα □