

$$= (F_1, F_2, F_3)(x, y, z) = (G_1, G_2, G_3)(x, y, z)$$

Έστω διανυσματικά πεδία  $\vec{F}(x, y, z)$ ,  $\vec{G}(x, y, z)$  και  
βαθμωτά πεδία  $f(x, y, z)$ ,  $g(x, y, z)$ , παραγωγίσιμα.

Να δείξουν τα ακόλουθα:

$$(i) \nabla(f+g) = \nabla f + \nabla g, \quad \nabla(fg) = f\nabla g + g\nabla f$$

Είναι:

$$\nabla(f+g) = \left( \frac{\partial}{\partial x}(f+g), \frac{\partial}{\partial y}(f+g), \frac{\partial}{\partial z}(f+g) \right)$$

$$= \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z} \right) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$+ \left( \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) = \nabla f + \nabla g.$$

$$\nabla(fg) = \left( \frac{\partial}{\partial x}(fg), \frac{\partial}{\partial y}(fg), \frac{\partial}{\partial z}(fg) \right)$$

$$= \left( g \frac{\partial f}{\partial x} + f \frac{\partial g}{\partial x}, g \frac{\partial f}{\partial y} + f \frac{\partial g}{\partial y}, g \frac{\partial f}{\partial z} + f \frac{\partial g}{\partial z} \right)$$

$$= g \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) + f \left( \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right)$$

$$= g \nabla f + f \nabla g.$$

$$\text{ii) } \nabla \cdot (\bar{F} + \bar{G}) = \nabla \cdot \bar{F} + \nabla \cdot \bar{G}$$

Eivou,

$$\nabla \cdot (\bar{F} + \bar{G}) = \frac{d}{dx} (\bar{F} + \bar{G})_1 + \frac{d}{dy} (\bar{F} + \bar{G})_2 + \frac{d}{dz} (\bar{F} + \bar{G})_3 = \frac{d}{dx} F_1 + \frac{d}{dx} G_1 + \frac{d}{dy} F_2 + \frac{d}{dy} G_2 + \frac{d}{dz} F_3 + \frac{d}{dz} G_3 = \nabla \cdot \bar{F} + \nabla \cdot \bar{G}$$


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$$\text{iii) } \nabla \times (\bar{F} + \bar{G}) = \nabla \times \bar{F} + \nabla \times \bar{G}$$

Eivou,

$$\nabla \times (\bar{F} + \bar{G}) = \overline{\det} \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ d/dx & d/dy & d/dz \\ (\bar{F} + \bar{G})_1 & (\bar{F} + \bar{G})_2 & (\bar{F} + \bar{G})_3 \end{pmatrix}$$

$$= \overline{\det} \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ d/dx & d/dy & d/dz \\ F_1 & F_2 & F_3 \end{pmatrix} + \overline{\det} \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ d/dx & d/dy & d/dz \\ G_1 & G_2 & G_3 \end{pmatrix}$$

$$= \nabla \times \bar{F} + \nabla \times \bar{G}, \quad \mu \in \{e_1, e_2, e_3\} \text{ συνιστούν βάση του } \mathbb{R}^3$$

$$iv) \nabla \cdot (f \vec{F}) = (\nabla f) \cdot \vec{F} + f (\nabla \cdot \vec{F})$$

Einvau, 
$$\nabla \cdot (f \vec{F}) = \frac{d}{dx} (f F_1) + \frac{d}{dy} (f F_2) + \frac{d}{dz} (f F_3)$$

$$= \frac{df}{dx} F_1 + \frac{dF_1}{dx} f + \frac{df}{dy} F_2 + \frac{dF_2}{dy} f + \frac{df}{dz} F_3 + \frac{dF_3}{dz} f$$

$$= \left( \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right) \cdot (F_1, F_2, F_3) + f \left( \frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz} \right)$$

$$= (\nabla f) \cdot \vec{F} + f (\nabla \cdot \vec{F}).$$


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$$v) \nabla \times (f \vec{F}) = (\nabla f) \times \vec{F} + f (\nabla \times \vec{F}).$$

Einvau, 
$$\nabla \times (f \vec{F}) = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f F_1 & f F_2 & f F_3 \end{pmatrix}$$

$$= \left( \frac{d}{dy} (f F_3) - \frac{d}{dz} (f F_2), \frac{d}{dz} (f F_1) - \frac{d}{dx} (f F_3), \frac{d}{dx} (f F_2) - \frac{d}{dy} (f F_1) \right)$$

$$\frac{d}{dy} (f F_3) = \left( \frac{df}{dy} F_3 + \frac{dF_3}{dy} f - \frac{df}{dz} F_2 - \frac{dF_2}{dz} f \right),$$

$$\frac{d}{dz} (f F_1) = \left( \frac{df}{dz} F_1 + \frac{dF_1}{dz} f - \frac{df}{dx} F_3 - \frac{dF_3}{dx} f, \frac{df}{dx} F_2 + \frac{dF_2}{dx} f - \frac{df}{dy} F_1 \right.$$

$$\left. - \frac{dF_1}{dy} f \right) = \left( \frac{df}{dy} F_3 - \frac{dF_2}{dz} f, \frac{df}{dz} F_1 - \frac{dF_3}{dx} f, \frac{df}{dx} F_2 - \frac{dF_1}{dy} f \right)$$

$$+ f \left( \frac{dF_3}{dy} - \frac{dF_2}{dz}, \frac{dF_1}{dz} - \frac{dF_3}{dx}, \frac{dF_2}{dx} - \frac{dF_1}{dy} \right)$$

$$= \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \frac{df}{dx} & \frac{df}{dy} & \frac{df}{dz} \\ F_1 & F_2 & F_3 \end{pmatrix} + f \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

$$= (\nabla f) \times \bar{F} + f (\nabla \times \bar{F}), \text{ όπου } \{\bar{e}_1, \bar{e}_2, \bar{e}_3\} \text{ η ορθή βάση του } \mathbb{R}^3$$

$$vii) \nabla \cdot (\bar{F} \times \bar{G}) = \bar{G} \cdot (\nabla \times \bar{F}) - \bar{F} \cdot (\nabla \times \bar{G})$$

$$\underline{\text{Είπαμε}} \quad \nabla \cdot (\bar{F} \times \bar{G}) = \det \begin{pmatrix} \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{pmatrix}$$

$$= \frac{d}{dx} (F_2 G_3) - \frac{d}{dx} (F_3 G_2) + \frac{d}{dy} (F_3 G_1) - \frac{d}{dy} (F_1 G_3) + \frac{d}{dz} (F_1 G_2)$$

$$- \frac{d}{dz} (F_2 G_1) = \frac{dF_2}{dx} G_3 + \frac{dG_3}{dx} F_2 - \frac{dF_3}{dx} G_2 - \frac{dG_2}{dx} F_3$$

$$+ \frac{dF_3}{dy} G_1 + \frac{dG_1}{dy} F_3 - \frac{dF_1}{dy} G_3 - \frac{dG_3}{dy} F_1 + \frac{dF_1}{dz} G_2$$

$$+ \frac{dG_2}{dz} F_1 - \frac{dF_2}{dz} G_1 - \frac{dG_1}{dz} F_2 = [ (G_1, G_2, G_3) \cdot$$

$$\left( \frac{dF_3}{dy} - \frac{dF_2}{dz}, \frac{dF_1}{dz} - \frac{dF_3}{dx}, \frac{dF_2}{dx} - \frac{dF_1}{dy} \right) ] - [ (F_1, F_2, F_3) \cdot$$

$$\left( \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}, \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}, \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right)$$

$$= \bar{G}_1 \cdot \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ d/dx & d/dy & d/dz \\ F_1 & F_2 & F_3 \end{pmatrix} = \bar{F} \cdot \det \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ d/dx & d/dy & d/dz \\ G_1 & G_2 & G_3 \end{pmatrix}$$

$$= \bar{G}_1 \cdot (\nabla \times \bar{F}) - \bar{F} \cdot (\nabla \times \bar{G}_1), \text{ όπου } \{\bar{e}_1, \bar{e}_2, \bar{e}_3\} \text{ η συνήθως}$$

Βάση του  $\mathbb{R}^3$

$$\text{vii) } \nabla \times (\bar{F} \times \bar{G}_1) = (\bar{G}_1 \cdot \nabla) \bar{F} - \bar{G}_1 (\nabla \cdot \bar{F}) - (\bar{F} \cdot \nabla) \bar{G}_1 + \bar{F} (\nabla \cdot \bar{G}_1)$$

$$\underline{\text{Είρα}}, \bar{F} \times \bar{G}_1 = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = (F_2 G_3 - F_3 G_2, F_3 G_1 - F_1 G_3, F_1 G_2 - F_2 G_1)$$

$$\text{, όπου } \nabla \times (\bar{F} \times \bar{G}_1) = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 G_3 - F_3 G_2 & F_3 G_1 - F_1 G_3 & F_1 G_2 - F_2 G_1 \end{vmatrix}$$

$$= \left( \frac{\partial F_1}{\partial y} G_2 + \frac{\partial G_2}{\partial y} F_1 - \frac{\partial F_2}{\partial y} G_1 - \frac{\partial G_1}{\partial y} F_2 - \frac{\partial F_3}{\partial z} G_1 - \frac{\partial G_1}{\partial z} F_3 \right.$$

$$+ \frac{\partial F_1}{\partial z} G_3 + \frac{\partial G_3}{\partial z} F_1, \frac{\partial F_2}{\partial z} G_3 + \frac{\partial G_3}{\partial z} F_2 - \frac{\partial F_3}{\partial z} G_2 - \frac{\partial G_2}{\partial z} F_3$$

$$\left. - \frac{\partial F_1}{\partial x} G_2 - \frac{\partial G_2}{\partial x} F_1 + \frac{\partial F_2}{\partial x} G_1 + \frac{\partial G_1}{\partial x} F_2, \frac{\partial F_3}{\partial x} G_1 + \frac{\partial G_1}{\partial x} F_3 \right)$$

$$\begin{aligned}
 & - \frac{dF_1}{dx} G_3 - \frac{dG_3}{dx} F_1 - \frac{dF_2}{dy} G_3 - \frac{dG_3}{dy} F_2 + \frac{dF_3}{dy} G_2 \\
 & + \frac{dG_2}{dy} F_3 \Big) \quad (1). \quad \text{Αναπτύσσω τώρα το 2ο μέλος της}
 \end{aligned}$$

σητούμενης σχέσης, κι έχω:

$$\begin{aligned}
 & (\bar{G}_1 \cdot \nabla) \bar{F} - \bar{G}_1 (\nabla \cdot \bar{F}) - (\bar{F} \cdot \nabla) \bar{G}_1 + \bar{F} (\nabla \cdot \bar{G}_1) = \\
 & = \left[ (G_1, G_2, G_3) \cdot \left( \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right) \right] (F_1, F_2, F_3) - (G_1, G_2, G_3) \left[ \right. \\
 & \left. \left( \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right) \cdot (F_1, F_2, F_3) \right] - \left[ (F_1, F_2, F_3) \cdot \left( \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right) \right] (G_1, G_2, G_3) \\
 & + (F_1, F_2, F_3) \left[ \left( \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right) \cdot (G_1, G_2, G_3) \right] =
 \end{aligned}$$

$$\begin{aligned}
 & \left( G_1 \frac{d}{dx} + G_2 \frac{d}{dy} + G_3 \frac{d}{dz} \right) (F_1, F_2, F_3) - (G_1, G_2, G_3) \left( \frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz} \right) \\
 & - \left( F_1 \frac{d}{dx} + F_2 \frac{d}{dy} + F_3 \frac{d}{dz} \right) (G_1, G_2, G_3) + (F_1, F_2, F_3) \left( \frac{dG_1}{dx} + \frac{dG_2}{dy} + \frac{dG_3}{dz} \right) \\
 & = \left( \cancel{G_1 \frac{dF_1}{dx}} + G_2 \frac{dF_1}{dy} + G_3 \frac{dF_1}{dz} - \cancel{G_1 \frac{dF_1}{dx}} - G_1 \frac{dF_2}{dy} - G_1 \frac{dF_3}{dz} \right. \\
 & \left. - \cancel{F_1 \frac{dG_1}{dx}} - F_2 \frac{dG_1}{dy} - F_3 \frac{dG_1}{dz} + \cancel{F_1 \frac{dG_1}{dx}} + F_1 \frac{dG_2}{dy} + F_1 \frac{dG_3}{dz} \right) \\
 & \left( G_1 \frac{dF_2}{dx} + \cancel{G_2 \frac{dF_2}{dy}} + G_3 \frac{dF_2}{dz} - \cancel{G_2 \frac{dF_1}{dx}} - \cancel{G_2 \frac{dF_2}{dy}} - G_2 \frac{dF_3}{dz} \right) \\
 & - \left( \cancel{F_1 \frac{dG_2}{dx}} - \cancel{F_2 \frac{dG_2}{dy}} - F_3 \frac{dG_2}{dz} + \cancel{F_2 \frac{dG_1}{dx}} + \cancel{F_2 \frac{dG_2}{dy}} + F_2 \frac{dG_3}{dz} \right) \\
 & \left( G_1 \frac{dF_3}{dx} + G_2 \frac{dF_3}{dy} + \cancel{G_3 \frac{dF_3}{dz}} - \cancel{G_3 \frac{dF_1}{dx}} - \cancel{G_3 \frac{dF_2}{dy}} - \cancel{G_3 \frac{dF_3}{dz}} \right)
 \end{aligned}$$

$$-F_1 \frac{dG_3}{dx} - F_2 \frac{dG_3}{dy} - \cancel{F_3 \frac{dG_3}{dz}} + F_3 \frac{dG_1}{dx} + F_3 \frac{dG_2}{dy} + \cancel{F_3 \frac{dG_3}{dz}} \quad (2)$$

Από (1), (2) τα δύο μέλη είναι ίσα.  $\square$

$$\text{viii)} \nabla(\vec{F} \cdot \vec{G}) = (\vec{G} \cdot \nabla) \vec{F} + (\vec{F} \cdot \nabla) \vec{G} + \vec{G} \times (\nabla \times \vec{F}) + \vec{F} \times (\nabla \times \vec{G})$$

Είναι  $\nabla(\vec{F} \cdot \vec{G}) = \nabla((F_1, F_2, F_3) \cdot (G_1, G_2, G_3))$

$$= \nabla(F_1 G_1 + F_2 G_2 + F_3 G_3) = \left( \frac{\partial}{\partial x} (F_1 G_1 + F_2 G_2 + F_3 G_3), \right.$$

$$\left. \frac{\partial}{\partial y} (F_1 G_1 + F_2 G_2 + F_3 G_3), \frac{\partial}{\partial z} (F_1 G_1 + F_2 G_2 + F_3 G_3) \right)$$

$$= \left( \frac{\partial F_1}{\partial x} G_1 + \frac{\partial G_1}{\partial x} F_1 + \frac{\partial F_2}{\partial x} G_2 + \frac{\partial G_2}{\partial x} F_2 + \frac{\partial F_3}{\partial x} G_3 + \frac{\partial G_3}{\partial x} F_3, \right.$$

$$\frac{\partial F_1}{\partial y} G_1 + \frac{\partial G_1}{\partial y} F_1 + \frac{\partial F_2}{\partial y} G_2 + \frac{\partial G_2}{\partial y} F_2 + \frac{\partial F_3}{\partial y} G_3 + \frac{\partial G_3}{\partial y} F_3, \left. \frac{\partial F_1}{\partial z} G_1 + \frac{\partial G_1}{\partial z} F_1 + \frac{\partial F_2}{\partial z} G_2 + \frac{\partial G_2}{\partial z} F_2 + \frac{\partial F_3}{\partial z} G_3 + \frac{\partial G_3}{\partial z} F_3 \right) \quad (1)$$

Αναπτύσσω τώρα το δεύτερο μέλος της ζητούμενης ισότητας, κι έχω:

$$(\vec{G} \cdot \nabla) \vec{F} + (\vec{F} \cdot \nabla) \vec{G} + \vec{G} \times (\nabla \times \vec{F}) + \vec{F} \times (\nabla \times \vec{G}) =$$

$$\left[ (G_1, G_2, G_3) \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \right] (F_1, F_2, F_3) + \left[ (F_1, F_2, F_3) \cdot \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \right] (G_1, G_2, G_3) + (G_1, G_2, G_3) \times \overline{\det} \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{pmatrix} + (F_1, F_2, F_3) \times \overline{\det} \begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ G_1 & G_2 & G_3 \end{pmatrix} =$$

$$\left( G_1 \frac{\partial}{\partial x} + G_2 \frac{\partial}{\partial y} + G_3 \frac{\partial}{\partial z} \right) (F_1, F_2, F_3) + \left( F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z} \right)$$

$$(G_1, G_2, G_3) + (G_1, G_2, G_3) \times \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$+ (F_1, F_2, F_3) \times \left( \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}, \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}, \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right)$$

$$= \left( G_1 \frac{\partial F_1}{\partial x} + G_2 \frac{\partial F_1}{\partial y} + G_3 \frac{\partial F_1}{\partial z}, G_1 \frac{\partial F_2}{\partial x} + G_2 \frac{\partial F_2}{\partial y} + G_3 \frac{\partial F_2}{\partial z}, \right.$$

$$G_1 \frac{\partial F_3}{\partial x} + G_2 \frac{\partial F_3}{\partial y} + G_3 \frac{\partial F_3}{\partial z} \Big) + \left( F_1 \frac{\partial G_1}{\partial x} + F_2 \frac{\partial G_1}{\partial y} + F_3 \frac{\partial G_1}{\partial z}, \right.$$

$$F_1 \frac{\partial G_2}{\partial x} + F_2 \frac{\partial G_2}{\partial y} + F_3 \frac{\partial G_2}{\partial z}, F_1 \frac{\partial G_3}{\partial x} + F_2 \frac{\partial G_3}{\partial y} + F_3 \frac{\partial G_3}{\partial z} \Big)$$

$$+ \left( F_2 \frac{\partial G_2}{\partial x} - F_2 \frac{\partial G_1}{\partial y} - F_3 \frac{\partial G_1}{\partial z} + F_3 \frac{\partial G_3}{\partial x}, F_3 \frac{\partial G_3}{\partial y} - F_3 \frac{\partial G_2}{\partial z} \right.$$

$$- F_1 \frac{\partial G_2}{\partial x} + F_1 \frac{\partial G_1}{\partial y}, F_1 \frac{\partial G_1}{\partial z} - F_1 \frac{\partial G_3}{\partial x} - F_2 \frac{\partial G_3}{\partial y} + F_2 \frac{\partial G_2}{\partial z} \Big)$$

$$+ \left( G_2 \frac{\partial F_2}{\partial x} - G_2 \frac{\partial F_1}{\partial y} - G_3 \frac{\partial F_1}{\partial z} + G_3 \frac{\partial F_3}{\partial x}, G_3 \frac{\partial F_3}{\partial y} - G_3 \frac{\partial F_2}{\partial z} \right.$$

$$- G_1 \frac{\partial F_2}{\partial x} + G_1 \frac{\partial F_1}{\partial y}, G_1 \frac{\partial F_1}{\partial z} - G_1 \frac{\partial F_3}{\partial x} - G_2 \frac{\partial F_3}{\partial y} + G_2 \frac{\partial F_2}{\partial z} \Big)$$



$$\begin{aligned}
&= \left( G_1 \frac{dF_1}{dx} + \cancel{G_2 \frac{dF_1}{dy}} + \cancel{G_3 \frac{dF_1}{dz}} + F_1 \frac{dG_{11}}{dx} + \cancel{F_2 \frac{dG_{11}}{dy}} + \cancel{F_3 \frac{dG_{11}}{dz}} \right. \\
&+ \cancel{F_2 \frac{dG_2}{dx}} - \cancel{F_2 \frac{dG_{11}}{dy}} - \cancel{F_3 \frac{dG_{11}}{dz}} + F_3 \frac{dG_{13}}{dx} + G_2 \frac{dF_2}{dx} - \cancel{G_2 \frac{dF_1}{dy}} \\
&- \cancel{G_3 \frac{dF_1}{dz}} + G_3 \frac{dF_3}{dx}, \quad \cancel{G_1 \frac{dF_2}{dx}} + G_2 \frac{dF_2}{dy} + \cancel{G_3 \frac{dF_2}{dz}} + \cancel{F_1 \frac{dG_{12}}{dx}} \\
&+ F_2 \frac{dG_{12}}{dy} + \cancel{F_3 \frac{dG_{12}}{dz}} + F_3 \frac{dG_{13}}{dy} - \cancel{F_3 \frac{dG_{12}}{dz}} - \cancel{F_1 \frac{dG_{12}}{dx}} + F_1 \frac{dG_{11}}{dy} \\
&+ G_3 \frac{dF_3}{dy} - \cancel{G_3 \frac{dF_2}{dz}} - \cancel{G_1 \frac{dF_2}{dx}} + G_1 \frac{dF_1}{dy}, \quad \cancel{G_1 \frac{dF_3}{dx}} + \cancel{G_2 \frac{dF_3}{dy}} \\
&+ G_3 \frac{dF_3}{dz} + \cancel{F_1 \frac{dG_{13}}{dx}} + \cancel{F_2 \frac{dG_{13}}{dy}} + F_3 \frac{dG_{13}}{dz} + F_1 \frac{dG_{11}}{dz} - \cancel{F_1 \frac{dG_{13}}{dx}} \\
&- \cancel{F_2 \frac{dG_{13}}{dy}} + F_2 \frac{dG_{12}}{dz} + G_1 \frac{dF_1}{dz} - \cancel{G_1 \frac{dF_3}{dx}} - \cancel{G_2 \frac{dF_3}{dy}} + G_2 \frac{dF_2}{dz} \Big) \quad (2)
\end{aligned}$$

Από (1), (2) τα δύο μέλη είναι ίσα  $\square$